POLAR RECIPROCALS
WITH
RESPECT TO A PAIR OF TRIANGLES.

By

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It is the purpose of this paper to investigate some of the properties of polars with respect to a triangle. With respect to two lines 1 and 1', intersecting at A, the polar of any point P in the plane but not on 1 or 1' may be defined as the line which is the fourth harmonic of PA with respect to 1 and 1'. In terms of perspectivities we may say, "If three points A, B, C, on a line 1 are perspective with three points A', B', C', on a line 1', then the meets of the pairs of lines AB' and A'B, AC' and A'C, BC' and B'C, are on a line p which is also on the meet of 1 and 1'. The line p is called the polar of P with respect to 1 and 1'." With respect to a triangle#: "Let $P_1P_2P_3P_4$ be the vertices of a complete quadrangle and let D_{12}, D_{13}, D_{14} be the vertices of the diagonal triangle, D_{12} being on the side P_1P_2, D_{13} on the side P_1P_3 and D_{14} on the side P_1P_4. The diagonal triangle is perspective with each one of the triangles formed by a set of three of the vertices of the quadrangle, the center of perspectivity being in each case the fourth vertex. This gives rise

#Veblen and Young, "Projective Geometry", Vol. I, Pg. 46.

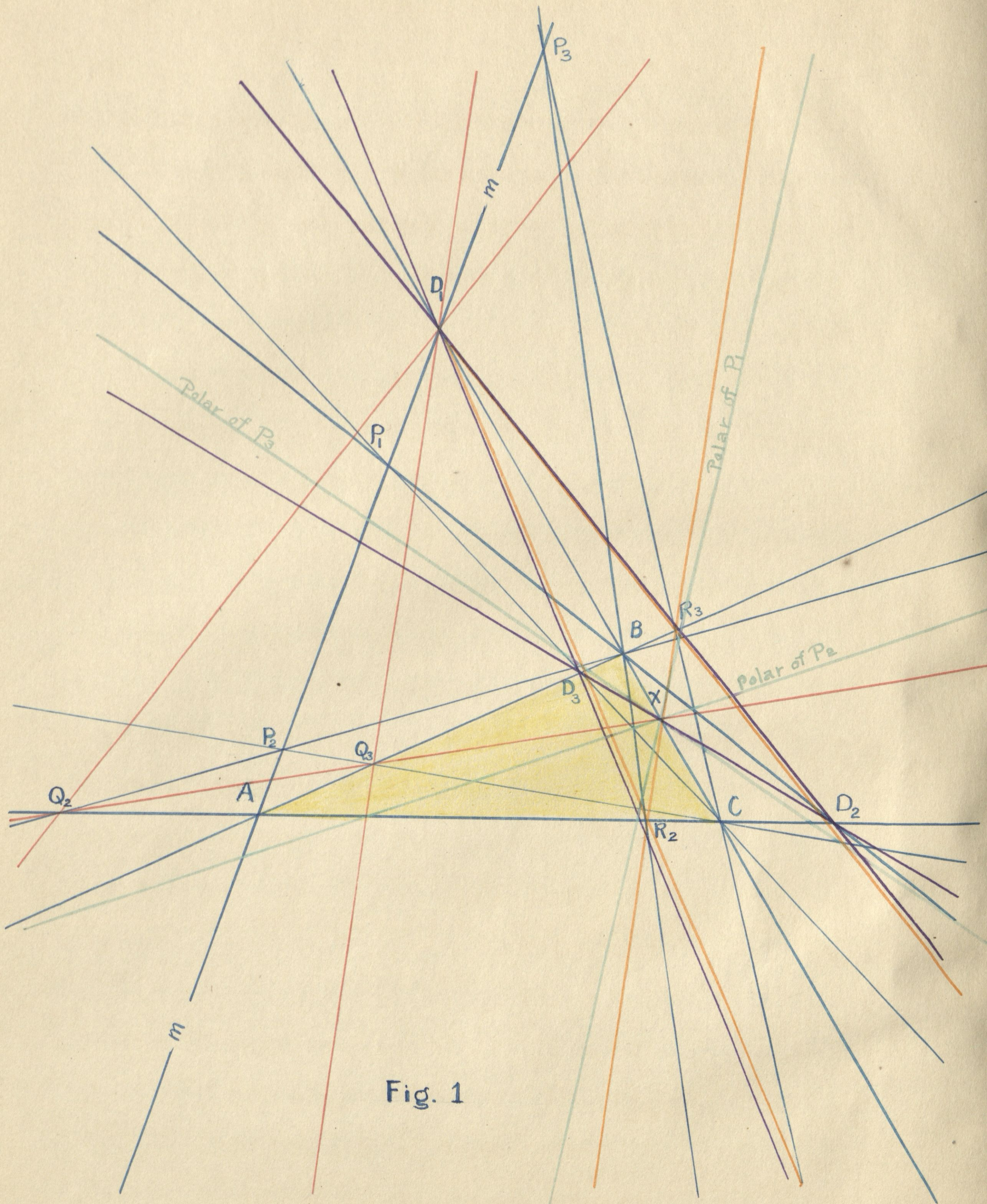

P3
m
D1
P1
Polar of P3
Polar of P1
R3
B
Polar of P2
D3
X
P2
Q3
Q2
A
C
D2
R2
m

Fig. 1

to four axes of perspectivity, one corresponding to each vertex of the quadrangle. The line thus uniquely associated with a vertex is called the polar of the point with respect to the triangle formed by the three remaining vertices."

THEOREM I. THE POLARS OF ALL POINTS ON A LINE THROUGH A VERTEX OF A TRIANGLE ARE ON A POINT ON THE SIDE OF THE TRIANGLE OPPOSITE THAT VERTEX.

(Figure I.)

Given: ABC with line m on A

To prove: polars of all points on m are on the same point X on BC.

Proof: Choose any three points P_1, P_2, P_3 on m and complete the quadrangles P_1ABC, P_2ABC, P_3ABC together with the three diagonal $\triangle$ $D_1D_2D_3$, $D_1Q_2Q_3$, $D_1R_2R_3$.

$D_1D_2D_3 \overset{P_1}{\barwedge} ABC$

$D_1Q_2Q_3 \overset{P_2}{\barwedge} ABC$

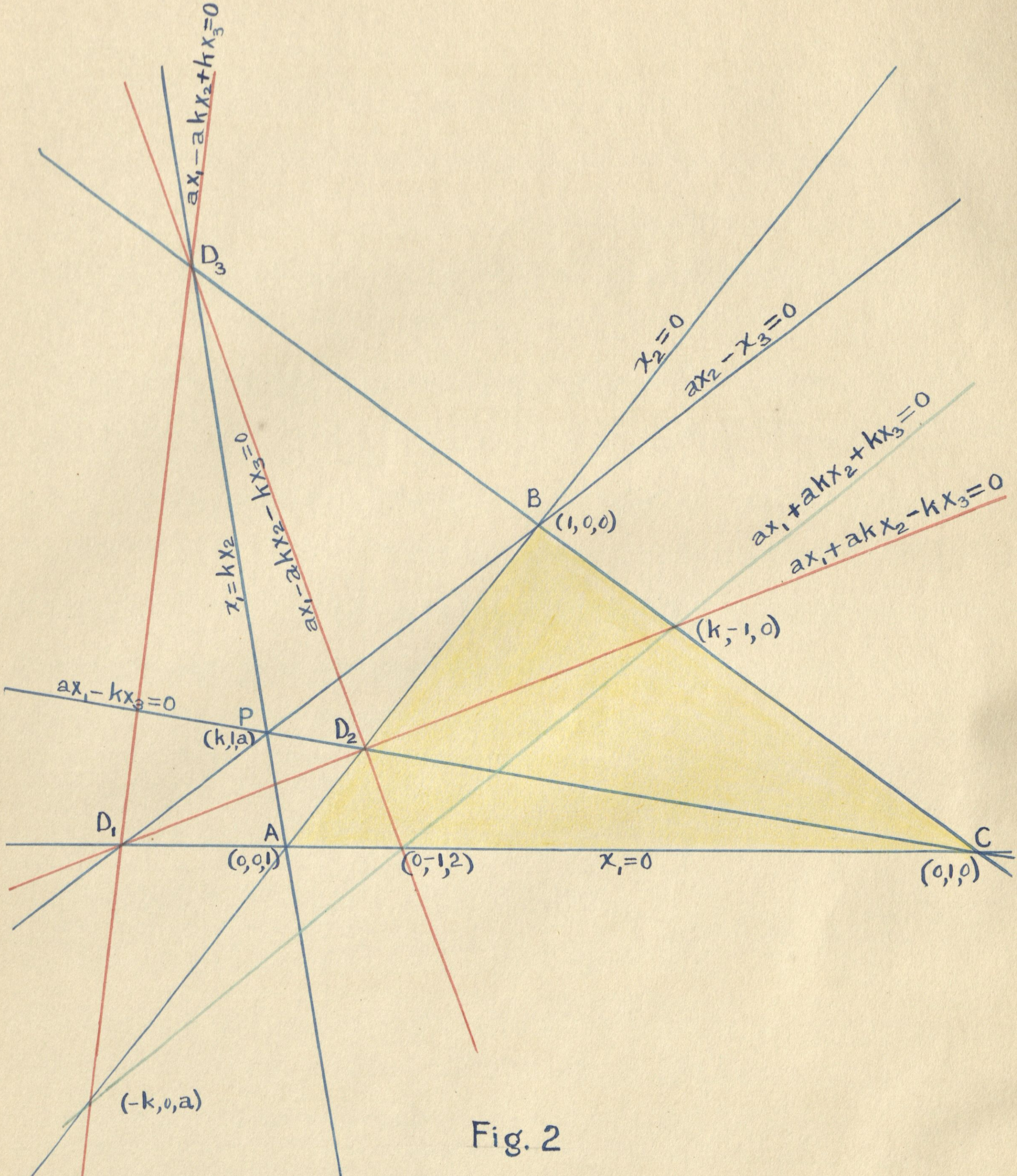
$D_1D_2D_3 = D_1Q_2Q_3$ (D_1 is self-corresponding)

Axis of perspectivity is BC

D_2D_3 and Q_2Q_3 meet on BC, say at X.

By same reasoning $D_1R_2R_3 \overset{BC}{\barwedge} D_1D_2D_3$

D_2D_3 meets R_2R_3 on BC, necessarily at X.



ax₁ − akx₂ + kx₃ = 0
D₃
x₂ = 0
ax₂ − x₃ = 0
ax₁ + akx₂ + kx₃ = 0
B
(1,0,0)
ax₁ + akx₂ − kx₃ = 0
x₁ = kx₂
ax₁ − akx₂ − kx₃ = 0
(k,−1,0)
ax₁ − kx₃ = 0
P
(k,1,a)
D₂
D₁
A
(0,0,1)
(0,−1,2)
x₁ = 0
C
(0,1,0)
(−k,0,a)

Fig. 2

By definition the three polars must be on X since in the three perspectivities QQ_{23}, DD_{23}, RR_{23} correspond to BC.

An analytic proof of the same theorem may be given as follows:

Let the three vertices of the fixed triangle be (in homogeneous coordinates).

(Figure II.)

$A \equiv (0,0,1)$

$B \equiv (1,0,0)$

$C \equiv (0,1,0)$

Then the homogeneous equations of the sides are

AB........$x_2 = 0$

AC........$x_1 = 0$

BC........$x_3 = 0$

and the equation of the triangle is

$$x_1 x_2 x_3 = 0$$

Any line through $A \equiv (0,0,1)$ may be written:

$x_1 = kx_2$

Any point P on $x_1 = kx_2$ may be designated $(k,1,a)$

Lines of quadrangle:

$$\left.\begin{array}{ll} \text{PB................} & ax_2 - x_3 = 0 \\ \text{AC................} & x_1 = 0 \end{array}\right\} \text{ opposite}$$

$$\left.\begin{array}{ll} PA \ldots\ldots\ldots\ldots\ldots\ldots & x_1 - kx_2 = 0 \\ BC \ldots\ldots\ldots\ldots\ldots\ldots & x_3 = 0 \end{array}\right\}$$

$$\left.\begin{array}{ll} PC \ldots\ldots\ldots\ldots\ldots\ldots & ax_1 - kx_3 = 0 \\ AB \ldots\ldots\ldots\ldots\ldots\ldots & x_2 = 0 \end{array}\right\}$$

Diagonal points: $D_1 \equiv (0,1,a)$

$D_2 \equiv (k,0,a)$

$D_3 \equiv (k,1,0)$

Diagonal lines: $D_1D_2 \ldots\ldots ax_1 + kax_2 - kx_3 = 0$

$D_1D_3 \ldots\ldots ax_1 - kax_2 + kx_3 = 0$

$D_2D_3 \ldots\ldots ax_1 - kax_2 - kx_3 = 0$

Points on polar: $(k,-1,0)$ intersection of D_1D_2 and BC

$(-1,0,a)$ " " D_1D_3 " AB

$(0,-1,a)$ " " D_2D_3 " A C

Polar: $ax_1 + akx_2 + kx_3 = 0$

This equation is always satisfied by $(k,-1,0)$

$\therefore$ all these polars are on $(k,-1,0)$ which is on $BC \equiv x_3 = 0$

Corollary: The polar figure of any triangle circumscribed about a fixed triangle is a triangle inscribed in the fixed triangle.

THEOREM I'. THE POLES OF ALL LINES ON A POINT P ON A SIDE AB OF A TRIANGLE ABC WITH REFERENCE TO ABC ARE ON A LINE ON C.

This is established by the principle of duality.

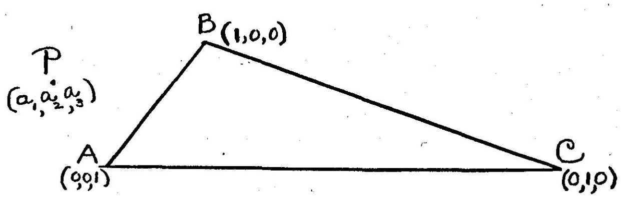
We may also define the polar# of a point (y_1,y_2,y_3) with respect to any function $F(x_1,x_2,x_3) = 0$ as

$$x_1\frac{\partial F}{\partial y_1} + x_2\frac{\partial F}{\partial y_2} + x_3\frac{\partial F}{\partial y_3} = 0$$

Where $\frac{\partial F}{\partial y_i}$ indicates that the coordinate y_i has been substituted in the partial derivative of the function with respect to x_i.

For the point $(k,1,a)$ we obtain, with reference to the function $x_1x_2x_3 = 0$ the polar: $ax_1 + akx_2 + kx_3 = 0$ which we have derived previously.

To prove that the polar so derived is identical with the polar derived by ordinary analytic methods let us write the polar of any point $P \equiv (a_1a_2a_3)$ by both methods, with reference to the triangle $x_1x_2x_3 = 0$.



Lines of quadrangle:

$$\left.\begin{array}{ll} PB\ldots\ldots & a_3x_2 - a_2x_3 = 0 \\ AC\ldots\ldots & x_1 = 0 \end{array}\right\}\text{opposite}$$

$$\left.\begin{array}{ll} PA\ldots\ldots & a_2x_1 - a_1x_2 = 0 \\ BC\ldots\ldots & x_2 = 0 \end{array}\right\}$$

#I.e. "a first polar". C. F. Grace and Young, "Algebra of Invariants", Cambridge, 1903, Pg. 41.

$$\left.\begin{array}{ll} PC \ldots\ldots & a_3x_1 - a_1x_3 = 0 \\ AB \ldots\ldots & x_2 = 0 \end{array}\right\}$$

Diagonal points: $D_1 \equiv (0, a_2, a_3)$

$D_2 \equiv (a_1, a_2, 0)$

$D_3 \equiv (a_1, 0, a_3)$

Diagonal lines: $D_1D_2 \ldots\ldots a_2a_3x_1 + a_1a_3x_2 - a_1a_2x_3 = 0$

$D_1D_3 \ldots\ldots a_2a_3x_1 - a_1a_3x_2 + a_1a_2x_3 = 0$

$D_2D_3 \ldots\ldots a_2a_3x_1 - a_1a_3x_2 - a_1a_2x_3 = 0$

Points on polar: $(-a_1, a_2, 0)$ intersection of D_1D_2 and BC

$(-a_1, 0, a_3)$ " " D_1D_3 " AB

$(0, -a_2, a_3)$ " " D_2D_3 " AC

Polar: $a_2a_3x_1 + a_1a_3x_2 + a_1a_2x_3 = 0$. By the operator $x_1\frac{\partial}{\partial y_1} + x_2\frac{\partial}{\partial y_2} + x_3\frac{\partial}{\partial y_3}$ we obtain the same equation.

Let us now write the polar reciprocal of any point, (y_1, y_2, y_3) with respect to the pair of triangles $x_1x_2x_3 = 0$ and $l'l''l''' = 0$ where

$$l' \equiv a_1u_1 + a_2u_2 + a_3u_3 = 0$$

$$l'' \equiv b_1u_1 + b_2u_2 + b_3u_3 = 0$$

$$l''' \equiv c_1u_1 + c_2u_2 + c_3u_3 = 0$$

Expanded, the equation of the latter triangle becomes

$$a_1b_1c_1u_1^3 + a_2b_2c_2u_2^3 + a_3b_3c_3u_3^3 + (a_1b_1c_2 - a_1b_2c_1 - a_2b_1c_1)u_1^2u_2 +$$
$$(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)u_1^2u_3 + (a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)u_1u_2^2 +$$
$$(a_1b_2c_3 + a_1b_3c_2 + a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 + a_3b_2c_1)u_1u_2u_3 +$$

$(a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)u_2^2u_3 + (a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)u_1u_3^2 +$

$(a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)u_2u_3^2 = 0$

The polar of (y_1, y_2, y_3) with respect to $x_1x_2x_3 = 0$ is

$$y_2y_3x_1 + y_1y_3x_2 + y_1y_2x_3 = 0$$

The pole of $[y_2y_3,\ y_1y_3,\ y_1y_2]$ with respect to 1'1''1''' $= 0$ is $u_1\Big[3a_1b_1c_1u_1^2 + 2(a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)u_1u_2 + 2(a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)u_1u_3 + (a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)u_2^2 + \sum_{i,j,k=1}^{i,j,k=3} a_ib_jc_ku_2u_3 + (a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)u_3^2\Big] + u_2\Big[3a_2b_2c_2u_2^2 + (a_1b_1c_2 + a_1b_2c_1 + a_2b_1c_1)u_1^2 + 2(a_1b_2c_2 + a_2b_1c_2 + a_2b_2c_1)u_1u_2 + 2(a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)u_2u_3 + \sum_{i,j,k=1}^{i,j,k=3} a_ib_jc_ku_1u_3 + (a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)u_3^2\Big] + u_3\Big[3a_3b_3c_3u_3^2 + (a_1b_1c_3 + a_1b_3c_1 + a_3b_1c_1)u_1^2 + (a_2b_2c_3 + a_2b_3c_2 + a_3b_2c_2)u_2^2 + \sum_{i,j,k=1}^{i,j,k=3} a_ib_jc_ku_1u_2 + 2(a_3b_3c_1 + a_3b_1c_3 + a_1b_3c_3)u_1u_3 + 2(a_3b_3c_2 + a_3b_2c_3 + a_2b_3c_3)u_2u_3\Big] = 0$

Writing this in different notation and substituting y_2y_3, y_1y_3, y_1y_2 for u_1, u_2, u_3 respectively, in the partials we have

$u_1\Big[3M_{111}y_2^2y_3^2 + 2(M_{112} + M_{121} + M_{211})y_1y_2y_3^2 + 2(M_{113} + M_{131} + M_{311})y_1y_2^2y_3 + (M_{122} + M_{212} + M_{221})y_1^2y_3^2 + \sum_{i,k,k=1}^{i,j,k=3} M_{ijk}y_1^2y_2y_3 + (M_{331} + M_{313} + M_{133})y_1^2y_2^2\Big] + u_2\Big[3M_{222}y_1^2y_3^2 + (M_{112} + M_{121} + M_{211})y_2^2y_3^2 + 2(M_{122} + M_{212} + M_{221})y_1y_2y_3^2 + 2(M_{223} + M_{232} + M_{322})y_1^2y_2y_3 + \sum_{i,j,k=1}^{i,j,k=3} M_{ijk}y_1y_2^2y_3 + (M_{332} + M_{323} + M_{233})y_1^2y_2^2\Big] + u_3\Big[3M_{333}y_1^2y_2^2 + (M_{113} + M_{131} + M_{311})y_2^2y_3^2 + (M_{223} + M_{232} + M_{322})y_1^2y_3^2 + \sum_{i,j,k=1}^{i,j,k=3} M_{ijk}y_1y_2y_3^2 + 2(M_{331} + M_{313} + M_{133})y_1y_2^2y_3 + 2(M_{332} + M_{323} + M_{233})y_1^2y_2y_3\Big] = 0$

Collecting all terms containing $y_2^2y_3^2$ first, then those containing $y_1^2y_2^2$ etc.,

$$y_2^2y_3^2\left[3M_{111}u_1 + (M_{112}+M_{121}+M_{211})u_2 + (M_{113}+M_{131}+M_{311})u_3\right] + y_1^2y_2^2\left[(M_{331}+\right.$$
$$\left.M_{313}+M_{133})u_1 + (M_{332}+M_{323}+M_{233})u_2 + 3M_{333}u_3\right] + y_1^2y_3^2\left[(M_{122}+M_{212}+M_{221})u_1\right.$$
$$\left.+\ 3M_{222}u_2 + (M_{223}+M_{232}+M_{322})u_3\right] + y_1y_2y_3^2\left[(2M_{112}+2M_{121}+2M_{211})u_1 +\right.$$
$$\left.(2M_{122}+2M_{212}+2M_{221})u_2 + (M_{123}+M_{132}+M_{213}+M_{231}+M_{312}+M_{321})u_3\right] +$$
$$y_1y_2^2y_3\left[(2M_{113}+2M_{131}+2M_{311})u_1 + (2M_{331}+2M_{313}+2M_{133})\ u_3 + (M_{123}+M_{132}+\right.$$
$$\left.M_{213}+M_{231}+M_{312}+M_{321})u_2\right] + y_1^2y_2y_3\left[(M_{123}+M_{132}+M_{213}+M_{231}+M_{312}+M_{321})u_1\ +\right.$$
$$\left.(2M_{223}+2M_{232}+2M_{322})u_2 + (2M_{332}+2M_{323}+2M_{233})u_3\right] = 0$$

Factoring:

$$y_2^2y_3^2a_1b_1(c_1u_1 + c_2u_2 + c_3u_3) + y_2^2y_3^2a_1c_1(b_1u_1 + b_2u_2 + b_3u_3) + y_2^2y_3^2b_1c_1(a_1u_1 +$$
$$a_2u_2 + a_3u_3) + y_1^2y_2^2a_3b_3(c_1u_1 + c_2u_2 + c_3u_3) + y_1^2y_2^2a_3c_3(b_1u_1 + b_2u_2 + b_3u_3) +$$
$$y_1^2y_2^2b_3c_3(a_1u_1 + a_2u_2 + a_3u_3) + y_1^2y_3^2a_2b_2(c_1u_1 + c_2u_2 + c_3u_3) + y_1^2y_3^2a_2c_2(b_1u_1 +$$
$$b_2u_2 + b_3u_3) + y_1^2y_3^2b_2c_2(a_1u_1 + a_2u_2 + a_3u_3) + y_1y_2y_3^2a_1b_2(c_1u_1 + c_2u_2 + c_3u_3) +$$
$$y_1y_2y_3^2a_1c_2(b_1u_1 + b_2u_2 + b_3u_3) + y_1y_2y_3^2b_1c_2(a_1u_1 + a_2u_2 + a_3u_3) + y_1y_2y_3^2a_2b_1(c_1u_1 +$$
$$c_2u_2 + c_3u_3) + y_1y_2y_3^2a_2c_1(b_1u_1 + b_2u_2 + b_3u_3) + y_1y_2y_3^2b_2c_1(a_1u_1 + a_2u_2 + a_3u_3) +$$
$$y_1y_2^2y_3a_1b_3(c_1u_1 + c_2u_2 + c_3u_3) + y_1y_2^2y_3a_1c_3(b_1u_1 + b_2u_2 + b_3u_3) + y_1y_2^2y_3b_1c_3(a_1u_1 +$$
$$a_2u_2 + a_3u_3) + y_1y_2^2y_3a_3b_1(c_1u_1 + c_2u_2 + c_3u_3) + y_1y_2^2y_3a_3c_1(b_1u_1 + b_2u_2 + b_3u_3) +$$
$$y_1y_2^2y_3b_3c_1(a_1u_1 + a_2u_2 + a_3u_3) + y_1^2y_2y_3a_2b_3(c_1u_1 + c_2u_2 + c_3u_3) + y_1^2y_2y_3a_2c_3(b_1u_1 +$$
$$b_2u_2 + b_3u_3) + y_1^2y_2y_3b_2c_3(a_1u_1 + a_2u_2 + a_3u_3) + y_1^2y_2y_3a_3b_2(c_1u_1 + c_2u_2 + c_3u_3) +$$
$$y_1^2y_2y_3a_3c_2(b_1u_1 + b_2u_2 + b_3u_3) + y_1^2y_2y_3b_3c_2(a_1u_1 + a_2u_2 + a_3u_3) = 0$$

Let $c_1u_1 + c_2u_2 + c_3u_3 = \gamma$

$b_1u_1 + b_2u_2 + b_3u_3 = \beta$

$a_1u_1 + a_2u_2 + a_3u_3 = \alpha$

$$\gamma\left[y_2^2y_3^2a_1b_1 + y_1^2y_2^2a_3b_3 + y_1^2y_3^2a_2b_2 + y_1y_2y_3^2(a_1b_2 + a_2b_1) + y_1y_2^2y_3(a_1b_3 + a_3b_1) +\right.$$

$y_1^2 y_2 y_3(a_2b_3 + a_3b_2)\Big] + \beta\Big[y_2^2y_3^2a_1c_1 + y_1^2y_2^2a_3c_3 + y_1^2y_3^2a_2c_2 + y_1y_2y_3^2(a_1c_2 + a_2c_1) +$
$y_1y_2^2y_3(a_1c_3 + a_3c_1) + y_1^2y_2y_3(a_2c_3 + a_3c_2)\Big] + \alpha\Big[y_2^2y_3^2b_1c_1 + y_1^2y_2^2b_3c_3 + y_1^2y_3^2b_2c_2$
$+ y_1y_2y_3^2(b_1c_2 + b_2c_1) + y_1y_2^2y_3(b_1c_3 + b_3c_1) + y_1^2y_2y_3(b_2c_3 + b_3c_2)\Big] = 0$

$\gamma\, y_2y_3a_1\Big[y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3\Big] + \gamma\, y_1y_3a_2\Big[y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3\Big] + \gamma\, y_1y_2a_3$
$\Big[y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3\Big] + \beta\, y_2y_3a_1\Big[y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3\Big] + \beta\, y_1y_3a_2\Big[y_2y_3c_1$
$+ y_1y_3c_2 + y_1y_2c_3\Big] + \beta\, y_1y_2a_3\Big[y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3\Big] + \alpha\, y_2y_3b_1\Big[y_2y_3c_1 + y_1y_3c_2$
$+ y_1y_2c_3\Big] + \alpha\, y_1y_3b_2\Big[y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3\Big] + \alpha\, y_1y_2b_3\Big[y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3\Big]$
$= 0$

$\gamma\,(y_2y_3a_1 + y_1y_3a_2 + y_1y_2a_3)\ (y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3) + \beta(y_2y_3a_1 + y_1y_3a_2 +$
$y_1y_2a_3)\ (y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3) + \alpha((y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3)\ (y_2y_3c_1 +$
$y_1y_3c_2 + y_1y_2c_3) = 0$

Let $y_2y_3a_1 + y_1y_3a_2 + y_1y_2a_3 = A$

$y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3 = B$

$y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3 = C$ and we have

$AB\gamma + AC\beta + BC\alpha = 0$ or, dividing by ABC,

$$\frac{\alpha}{A} + \frac{\beta}{B} + \frac{\gamma}{C} = 0$$

Collecting coefficients of u_1, u_2, u_3, we have $(ABc_1 +$
$ACb_1 + BCa_1)u_1 + (ABc_2 + ACb_2 + BCa_2)u_2 + (ABc_3 + ACb_3 + BCa_3)u_3$
$= 0$

Dividing by ABC, we have

$$\left(\frac{c_1}{C} + \frac{b_1}{B} + \frac{a_1}{A}\right)u_1 + \left(\frac{c_2}{C} + \frac{b_2}{B} + \frac{a_2}{A}\right)u_2$$

$$+ \left(\frac{c_3}{C} + \frac{b_3}{B} + \frac{a_3}{A}\right)u_3 = 0$$

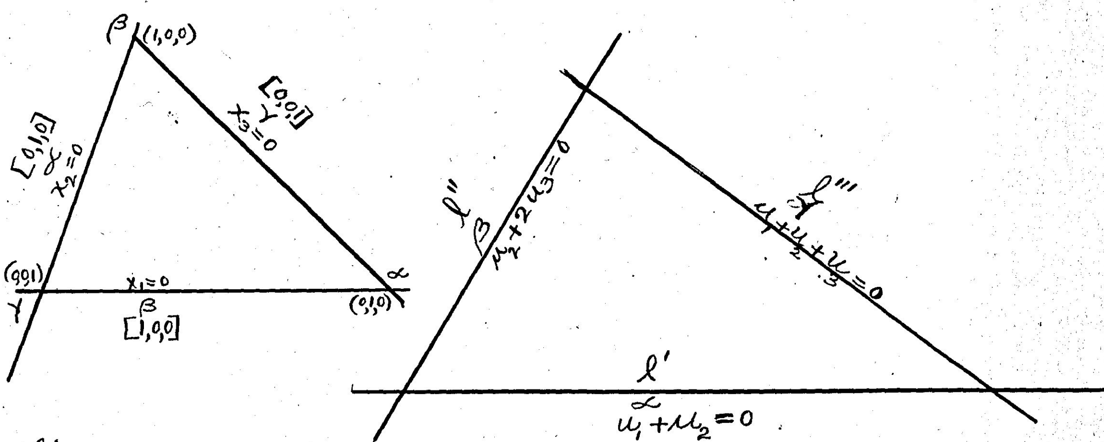
Thus by the process of writing the polar of $(y_1y_2y_3)$ with respect to $x_1x_2x_3 = 0$ and the pole of that polar with respect to $1'1''1''' = 0$ we have the point to point transformation:

$$y_1 \rightarrow \frac{a_1}{A} + \frac{b_1}{B} + \frac{c_1}{C}$$

$$y_2 \rightarrow \frac{a_2}{A} + \frac{b_2}{B} + \frac{c_2}{C}$$

$$y_3 \rightarrow \frac{a_3}{A} + \frac{b_3}{B} + \frac{c_3}{C}$$

Check on duality of process just defined. Polar of $(1,1,-1)$ with respect to $x_1x_2x_3 = 0$ is $-x_1 - x_2 + x_3 = 0$



$$\begin{array}{l} l', \alpha \\ l'', \beta \\ l''', \gamma \end{array} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1$$

Pole of $[-1,-1,1]$ with respect to $1'1''1''' = 0$ is

$(u_1 + u_2 + u_3)\ (-1-1)\ (-1 + 2) + (u_2 + 2u_3)\ (-2)\ (-1-1+1) + (u_1 + u_2)\ (1)\ (-1) = 0$

Collected, $-3u_1 - u_2 + 2u_3 = 0$

I. e. $(1,1-1) \rightarrow (-3,-1,2)$

To check back on this, i.e., to write the polar of $(-3,-1,2)$ with respect to $1'1''1''' = 0$ and the pole of that polar with respect to $x_1x_2x_3 = 0$ we will have to dualize the equations of the poles and polars. I. e., when the equation of a pole becomes the equation of a polar we must replace line coordinates with point coordinates, must replace the equation of a point in line coordinates with the equation of a line in point coordinates. Specifically, in the formula for the pole of $[y_2y_3,\ y_1y_3,\ y_1y_2]$ with respect to $1'1''1''' = 0$, which is: $1'''(y_2y_3a_1 + y_1y_3a_2 + y_1y_2a_3)\ (y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3) + 1''(\quad)\ (\quad) + 1'(\quad)\ (\quad) = 0$, we must put the equation of the family of lines on a point(said point to be the intersection of 1' and 1'') in place of 1''', the equation of the family of points on a line.

Thus, in place of 1''' we put $2x_1 - 2x_2 - x_3$

" " " 1'' " " $x_1 - 2x_2 + x_3$

" " " 1' " " $x_1 - x_2$

Accordingly, polar of $(-3,1,2)$ with respect to $1'1''1''' = 0$ is $(2x_1 - 2x_2 + x_3)\ (-3 + 1)\ (-3 + 2 + 2) + (x_1 - 2x_2 + x_3)\ (-3 + 1)\ (-6 + 2 + 2) + (x_1 - x_2)\ (-3 + 2 + 2)$ times

$(-6 + 2 + 2) = 0$

Collected $\qquad -x_1 - x_2 + x_3 = 0$

$$x_1 + x_2 - x_3 = 0$$

Now in place of $x_1 = 0$ we will put the equation of the opposite side $u_2 = 0$ etc. $u_1(y_2y_3) + u_2(y_1y_3) + u_3(y_1y_2) = 0$ becomes $u_1 + u_2 - u_3 = 0$ or $(-3,-1,2) \rightarrow (1,1,-1)$

THEOREM II. THE POLAR RECIPROCAL OF THE GENERAL STRAIGHT LINE $PX_1 + QX_2 + RX_3 = 0$, WITH RESPECT TO A PAIR OF TRIANGLES IS A CURVE OF THE FOURTH DEGREE.

By the transformation previously derived, we have the following as the equation of the reciprocal figure:

$$p\left[\frac{a_1}{A} + \frac{b_1}{B} + \frac{c_1}{C}\right] + q\left[\frac{a_2}{A} + \frac{b_2}{B} + \frac{c_2}{C}\right] + r\left[\frac{a_3}{A} + \frac{b_3}{B} + \frac{c_3}{C}\right] = 0$$

$$pa_1BC + qa_2BC + ra_3BC + pb_1AC + qb_2AC + rb_3AC + pc_1AB + qc_2AB + rc_3AB = 0$$

$$(pa_1 + qa_2 + ra_3)BC + (pb_1 + qb_2 + rb_3)AC + (pc_1 + qc_2 + rc_3)AB = 0$$

p, q, r, a_i, b_i, c_i, are all constants. We may therefore write the equation

$$k'BC + k''AC + k'''AB = 0$$

$$k'(y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3)(y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3) + k''(y_2y_3a_1 + y_1y_3a_2 + y_1y_2a_3)(y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3) + k'''(y_2y_3a_1 + y_1y_3a_2 + y_1y_2a_3)(y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3) = 0$$

$$y_2^2y_3^2(b_1c_1k' + a_1c_1k'' + a_1b_1k''') + y_1^2y_3^2(b_2c_2k' + a_2c_2k'' + a_2b_2k''') +$$
$$y_1^2y_2^2(b_3c_3k' + a_3c_3k'' + a_3b_3k''') + y_1y_2y_3^2(b_1c_2k' + a_1c_2k'' + a_1b_2k''')$$

$+$ etc. $= 0$, a homogenous equation of the fourth degree.

THEOREM III. THE POLAR RECIPROCAL OF THE PARABOLA, $X_2^2 = KX_1X_3$, IS A CURVE OF THE EIGTH DEGREE.

The equation of the reciprocal figure is:

$$\left[\frac{a_2}{A} + \frac{b_2}{B} + \frac{c_2}{C}\right]^2 = K\left[\frac{a_1}{A} + \frac{b_1}{B} + \frac{c_1}{C}\right]\left[\frac{a_3}{A} + \frac{b_3}{B} + \frac{c_3}{C}\right]$$

$$\frac{a^2}{A^2} + \frac{b^2}{B^2} + \frac{c^2}{C^2} + \frac{2a_2b_2}{AB} + \frac{2a_2c_2}{AC} + \frac{2b_2c_2}{BC} = k\left[\frac{a_1a_3}{A^2} + \frac{a_1b_3}{AB} + \frac{a_1c_3}{AC} + \right.$$
$$\left.\frac{a_3b_1}{AB} + \frac{b_1b_3}{B^2} + \frac{b_1c_3}{BC} + \frac{a_3c_1}{AC} + \frac{b_3c_1}{BC} + \frac{c_1c_3}{C^2}\right.$$

Collecting, in the right member,

$$\frac{a_2^2}{A^2} + \frac{b_2^2}{B^2} + \frac{c_2^2}{C^2} + \frac{2a_2b_2}{AB} + \frac{2a_2c_2}{AC} + \frac{2b_2c_2}{BC} = k\left[\frac{a_1a_3}{A^2} + \frac{a_1b_3 + a_3b_1}{AB} + \right.$$
$$\left.\frac{a_1c_3 + a_3c_1}{AC} + \frac{b_1c_3 + b_3c_1}{BC} + \frac{b_1b_3}{B^2} + \frac{c_1c_3}{C^2}\right]$$

Transposing,

$$\frac{a_2^2 - ka_1a_3}{A^2} + \frac{b_2^2 - kb_1b_3}{B^2} + \frac{c_2^2 - kc_1c_3}{C^2} + \frac{2a_2b_2 - (a_1b_3 + a_3b_1)k}{AB} +$$
$$\frac{2a_2c_2 - k(a_1c_3 + a_3c_1)}{AC} + \frac{2b_2c_2 - k(b_1c_3 + b_3c_1)}{BC} = 0$$

$(a_2^2 - ka_3a_1)B^2C^2 + (b_2^2 - kb_3b_1)A^2C^2 + (c_2^2 - kc_1c_3)A^2B^2 + \left[2a_2b_2 - k(a_1b_3 + a_3b_1)\right]ABC^2 + \left[2a_2c_2 - k(a_1c_3 + a_3c_1)\right]AB^2C + \left[2b_2c_2 - k(b_1c_3 + b_3c_1)\right]A^2BC = 0$

Since a_i, b_i, c_i, and k are all constants, we may write $k_1B^2C^2 + k_2A^2C^2 + k_3A^2B^2 + k_4ABC^2 + k_5AB^2C + k_6A^2BC = 0$, a homogeneous equation, the first term of which, without abridgment, is $k_1(y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3)^2(y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3)^2$. This term being clearly of the eighth degree, the theorem is proved.

THEOREM IV. THE POLAR RECIPROCAL OF THE GENERAL CONIC, $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, IS A CURVE OF THE EIGHTH DEGREE.

In homogenous coordinates, and with different constants, the equation is $Kx_1^2 + Lx_1x_2 + Mx_2^2 + Nx_1x_3 + Ox_2x_3 + Px_3^2 = 0$

The equation of the reciprocal figure is:

$$K\left[\frac{a_1}{A} + \frac{b_1}{B} + \frac{c_1}{C}\right]^2 + L\left[\frac{a_1}{A} + \frac{b_1}{B} + \frac{c_1}{C}\right]\left[\frac{a_2}{A} + \frac{b_2}{B} + \frac{c_2}{C}\right] + M\left[\frac{a_2}{A} + \frac{b_2}{B} + \frac{c_2}{C}\right]^2$$

$$+ N\left[\frac{a_1}{A} + \frac{b_1}{B} + \frac{c_1}{C}\right]\left[\frac{a_3}{A} + \frac{b_3}{B} + \frac{c_3}{C}\right] + O\left[\frac{a_2}{A} + \frac{b_2}{B} + \frac{c_2}{C}\right]\left[\frac{a_3}{A} + \frac{b_3}{B} + \frac{c_3}{C}\right] +$$

$$P\left[\frac{a_3}{A}+\frac{b_3}{B}+\frac{c_3}{C}\right]^2 = 0$$

Collecting all terms which have A^2 for a denominator, we have

$$\frac{Ka_1^2 + La_1a_2 + Ma_2^2 + Na_1a_3 + Oa_2a_3 + Pa_3^2}{A^2}$$

Since all numerators would consist of constants as in the term above, we may write the equation

$$\frac{k_1}{A^2}+\frac{k_2}{B^2}+\frac{k_3}{C^2}+\frac{k_4}{AB}+\frac{k_5}{AC}+\frac{k_6}{BC} = 0$$

$$k_1B^2C^2 + k_2A^2C^2 + k_3A^2B^2 + k_4ABC^2 + k_5AB^2C + k_6A^2BC = 0$$

Without abridgment, the first term is:

$$k_1(y_2y_3b_1 + y_1y_3b_2 + y_1y_2b_3)^2\,(y_2y_3c_1 + y_1y_3c_2 + y_1y_2c_3)^2$$

Sinc y_1, y_2, y_3 are the only variables, we have a term of eighth degree.

THEOREM V. THE POLAR RECIPROCAL OF THE SPECIAL CUBIC $y = x^3$, IS A CURVE OF THE TWELFTH DEGREE.

In homogeneous form the above cubic is $x_2x_3^2 = x_1^3$.

The equation of its polar reciprocal is

$$\left[\frac{a_2}{A}+\frac{b_2}{B}+\frac{c_2}{C}\right]\left[\frac{a_3}{A}+\frac{b_3}{B}+\frac{c_3}{C}\right]^2 = \left[\frac{a_1}{A}+\frac{b_1}{B}+\frac{c}{C}\right]^3$$

$$\left[\frac{a_2}{A}+\frac{b_2}{B}+\frac{c_2}{C}\right]\left[\frac{a_3^2}{A^2}+\frac{b_3^2}{B^2}+\frac{c_3^2}{C^2}+\frac{2a_3b_3}{AB}+\frac{2a_3c_3}{AC}+\frac{2b_3c_3}{BC}\right]=\frac{a_1^3}{A^3}+$$

$$\frac{b_1^3}{B^3}+\frac{c_1^3}{C^3}+\frac{3a_1^2}{A^2}\left[\frac{b_1}{B}+\frac{c_1}{C}\right]+\frac{3a_1}{A}\left[\frac{b_1^2}{B^2}+\frac{2b_1c_1}{BC}+\frac{c_1^2}{C^2}\right]+\frac{3b_1^2c_1}{BC}+\frac{3b_1c_1^2}{BC}.$$

$$\frac{a_2a_3^2}{A^3}+\frac{b_2b_3^2}{B^3}+\frac{c_2c_3^2}{C^3}+\text{etc.}=\frac{a_1^3}{A^3}+\frac{b_1^3}{B^3}+\text{etc.}$$

The L. C. D. is $A^3B^3C^3$. Clearing of fractions, we have $a_2a_3^2B^3C^3 + b_2b_3^2A^3C^3 + \text{etc.}$

B is of second degree in the variables y_1, y_2, y_3.

B^3 " " sixth " " " " y_1, y_2, y_3.

The term $a_2a_3^2B^3C^3$ is therefore of the twelfth degree.

THEOREM VI. THE POLAR RECIPROCAL OF THE GENERAL EQUATION OF THE NTH DEGREE IS A CURVE OF DEGREE 4N.

The general equation of nth degree in two unknowns, in homogeneous coordinates is $\sum K_{ijk}x_1^ix_2^jx_3^k = 0$ $(i + j + k = n)$ The equation of the reciprocal figure is

$$\sum K_{ijk}\left[\frac{a_1}{A}+\frac{b_1}{B}+\frac{c_1}{C}\right]^i\left[\frac{a_2}{A}+\frac{b_2}{B}+\frac{c_2}{C}\right]^j\left[\frac{a_3}{A}+\frac{b_3}{B}+\frac{c_3}{C}\right]^k=0\ (i+j+k$$

$= n)$. In the expansion of this all denominators would of nth degree. The L. C. D. is $\therefore A^nB^nC^n$ and in clearing of fractions we obtain terms of the form $kA^rB^sC^t$ where $r + s + t = 2n$, A, B, C are of second degree in y_1, y_2, y_3. Therefore $kA^rB^sC^t$ is of degree $2r + 2s + 2t$ in the variables y_1, y_2, y_3 or of degree $4n$.